

Denote by $\mathbb{N}$ the set of natural numbers, *i.e.*, the set of positive integers.

Exercise 1

- (1) Find all the sequences of real numbers $(d_n)_{n \in \mathbb{N}}$ satisfying the following recurrence equation

$$d_n + 2d_{n-1} + d_{n-2} = 0 \text{ for every } n \geq 3, n \in \mathbb{N}.$$

- (2) Let us consider the determinant of size $n \times n$, for $n \in \mathbb{N}$, defined by

$$D_n := \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{vmatrix}$$

Compute D_n for $n = 2, 3, 4$. Note that $D_1 = 0$.

- (3) Show that the sequence $(D_n)_{n \in \mathbb{N}}$ defined in (2) satisfies the recurrence equation in (1), *i.e.*, for all $n \in \mathbb{N}$, $n \geq 3$, we have $D_n + 2D_{n-1} + D_{n-2} = 0$.
- (4) Deduce a formula for D_n for every $n \in \mathbb{N}$.

Exercise 2

Consider the function f defined on the interval $[0, +\infty)$ by

$$f(x) := \begin{cases} 1 & \text{if } x = 0 \\ \frac{x}{\ln(1+x)} & \text{if } x \in (0, +\infty). \end{cases}$$

- (1) Determine the sign of f on the interval $[0, \infty)$. Deduce the existence of the sequence $(u_n)_{n \in \mathbb{N} \cup \{0\}}$ defined by

$$u_0 = e \quad \text{and} \quad \forall n \in \mathbb{N} \cup \{0\}, \quad u_{n+1} = f(u_n).$$

- (2) Show that f is continuous on the interval $[0, \infty)$.

- (3) Calculate the derivative of f on the interval $(0, +\infty)$, and show that f is of class C^1 on $(0, +\infty)$. (Recall that the class C^1 consists of all differentiable functions whose derivative is continuous.)
- (4) Give the Taylor expansion of order 2 of the function
- $$\ln(1+x) - \frac{x}{x+1}$$
- in a neighborhood of 0, and determine a polynomial asymptotics for $f'(x)$ when x tends to 0.
- (5) Show that f is of class C^1 on $[0, +\infty)$.
- (6) Prove that for all $x \geq e - 1$, we have
- $$f(x) \leq x \quad \text{and} \quad (x+1) \ln(x+1) \geq x+1.$$
- Deduce that for all $x \geq e - 1$, we have $f'(x) \geq 0$.
- (7) Show that for any $n \in \mathbb{N} \cup \{0\}$, we have $u_n \geq e - 1$.
- (8) Prove that the sequence u_n converges and determine the value of its limit.

Exercise 3

Consider the set $E = \{1, 2, 3, 4, 5, 6\}$. How many positive integer numbers can be formed by digits belonging to E in each of the following cases if, in every number, every digit can appear at most once ?

- (1) Numbers formed by 6 digits.
- (2) Numbers formed by 4 digits.
- (3) Numbers formed by 4 digits and having 1 as the first digit.
- (4) Numbers formed by 4 digits such that one of these digits is 1.
- (5) Numbers formed by 4 digits such that 1 and 3 are among these digits.